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An investigation of the Lines of Curvature of Ellipsoids, Hyperboloids and Paraboloids, with a view to the improvement of the Theory of the structure and decoration of Domes. By the Rev. Dionysius Lardner, *A. M. M. R. I. A.*

Read June 14, 1824.

THERE is no department of architecture in which the principles of geometry are more usefully or elegantly applied than in the theory of the structure and decoration of domes. A dome is a vault or roof, forming a segment of a curved surface, the concavity of which is turned inwards. All horizontal sections should be similar figures, and it should be supported by a base of the same figure. It is however by no means necessary that the edifice over which the dome is to be thrown should be of this figure, for there are methods well known to architects, but which would be foreign to my present object to enter upon, by which any polygonal base may be made to terminate in a circle, or in such other curve as may be required. From this curve a cylindrical wall may be erected, on which the dome is to be raised. Domes are denominated circular or elliptical, according to the base on which they are constructed, which is usually one of these figures. Domes, which rise higher than the semitransverse axis of the base, if they be elliptical, or than the radius of the base, if they be circular, are called *surmounted domes;* those whose altitude is less than the semiconjugate axis of the elliptic base, or than the radius of the circular base, are called *surbased domes* Elliptical domes of an intermediate altitude are called *mean elliptical* domes, and a circular dome, whose altitude is equal to the radius of the base, is called an *hemispherical dome.* When the base of the dome, and therefore all horizontal sections of it, are circles, it is a surface of revolution, and is commonly called *cupola.*

The lines of curvature of the domic surface are peculiarly calculated to suggest not only principles for the construction of the dome, but point out the most natural and elegant decorations for it ; decorations not depending on the fancy or caprice of the architect, but arising necessarily out of the form and properties of the surface itself.

The perpendiculars to the surface passing through the several points of a line of curvature, form a surface perpendicular to the surface of the dome. There are two systems of lines of curvature on every surface, and the lines of each system intersect those of the other system at right angles. If several lines of each system be described upon the domic surface, they will divide the whole vault into regular and symmetrical compartments, included by curvilinear limits intersecting perpendicularly. The perpendiculars to the surface through all these lines will form curved surfaces, which will also intersect perpendicularly, and will divide a shell of the vault of any assumed thickness into voussoirs bounded by those rectangular curved surfaces. If, however, the lines of curvature be drawn sufficiently close, so as to reduce the magnitude of these voussoirs to a sufficiently small proportion to the entire magnitude of the dome, the bounding surfaces may be considered as planes, scil. the tangent planes to the normal developable surfaces already mentioned. The magnitude of the voussoirs must be regulated by the magnitude of the whole vault, and the peculiar nature of the materials of which it is constructed. The decorations of the internal concave surface of the dome, as well as the external convex surface, should be all regulated by the directions of the lines of curvature, and by the compartments into which they resolve the surface. The position, directions, and general figure of these lines should be also attended to in the decoration and structure of the hall over which the dome is suspended. The columns, cornices, and the other ornamental parts of the building, both internal, and internal should harmonize with these lines.

There are in some surfaces points called umbilical points, which are so surrounded by the lines of curvature that they all turn their concavities towards them. On different sides of these points, therefore, the lines of both systems have their curvatures turned in opposite directions. In the decorations of domes having such points they should therefore be attended to, as they must immediately attract the eye from the peculiar disposition of the compartments of the dome which surround them. When such points occur in either of the principal vertical sections of the dome, and at any considerable elevation from its base, they would be adapted to receive any suspended ornament, such as lamps, lustres, chandeliers, &c. If these points, as is the case in the mean ellipsoidal dome, are placed upon the base, they suggest a peculiar disposition in the architectural decorations of the lower part of the hall.

The projections of the lines of curvature upon the floor of the hall point out the proper method of disposing its flagging, so as to correspond with the decorations already suggested for the other parts of the building. The methods of ornamenting an edifice, thus deduced necessarily from the very nature of the structure, while they set no limit either to the richness or the quantity of ornament, dispose it in a manner simple and graceful, and calculated to display the peculiar nature and construction of the edifice itself.

The most common species of dome is a cupola, but it is far from being that which admits of the richest decoration, or which produces the most pleasing effect. If the building be low, and the base elliptic, a surbased ellipsoidal dome is probably the most elegant. If it be more elevated, it is necessary that the dome should be surmounted, because the encreased elevation will necessarily diminish the apparent vertical height. The surfaces of domes are always however segments either of an ellipsoid, hyperboloid, or paraboloid. Of each of the latter surfaces there are two species, one only of which seems fitted for domes. I propose, in the following paper, to investigate the lines of curvature of these surfaces. Those of the ellipsoid have already been determined by Monge; however, as they can be all ultimately derived from the integration of the same equation, I shall include the ellipsoid to render the present investigation complete.

Of the lines of curvature of surfaces of the second degree.

The equation of those surfaces which have a centre related to the principal axes as axes of coordinates is

$$Az^2 + A'y^2 + A''x^2 + F = 0.$$

To determine the differential coefficients of the first and second order, let this equation be differentiated considering z as a function of xy; the results will be

$$\frac{dz}{dx} = -\frac{A''x}{Az}$$

$$\frac{dz}{dy} = -\frac{A'y}{Az}$$

$$\frac{d^2z}{dx^2} = \frac{A''}{A^2z^3}(A'y^2 + F)$$

$$\frac{d^2z}{dx\,dy} = -\frac{A'A''xy}{A^2z^3}$$

$$\frac{d^2z}{dy^2} = \frac{A'}{A^2z^3}(A''x^2 + F)$$

Those surfaces, which have no centre, are represented by the equation

$$A'y^2 + A''x^2 + 2\,C''z = 0.$$

The principal axis being the axis of z, and the plane of xy being a tangent plane through the vertex of the surface.

The differential coefficients of the first and second order are, in this case

$$\frac{dz}{dx} = -\frac{A''x}{C''}$$

$$\frac{dz}{dy} = -\frac{A'y}{C''}$$

$$\frac{d^2z}{dx^2} = -\frac{A''}{C''}$$

$$\frac{d^2z}{dydx} = 0.$$

$$\frac{d^2z}{dy^2} = -\frac{A'}{C''}$$

Let these five coefficients be expressed by p, q, r, s, t, and let their values be substituted in the equation

$$\left\{s(1+q^2)-pqt\right\}\cdot\frac{dy^2}{dx^2}+\left\{r(1+q^2)-t(1+p^2)\right\}\cdot\frac{dy}{dx}+pqr-s(1+p^2)=0.$$

(See Lacroix's Differential and Integral Calculus, 4to. Tom. I. p. 576, 2d edit.) The result of this substitution will, for central surfaces, be

$$\frac{dy^2}{dx^2}-\left(\frac{y}{x}-\frac{\mathrm{A}-\mathrm{A}''}{\mathrm{A}-\mathrm{A}'}\cdot\frac{x}{y}+\frac{\mathrm{A}(\mathrm{A}''-\mathrm{A}')\mathrm{F}}{\mathrm{A}'\mathrm{A}''(\mathrm{A}-\mathrm{A}')\,xy}\right)\frac{dy}{dx}-\frac{\mathrm{A}-\mathrm{A}''}{\mathrm{A}-\mathrm{A}'}=0$$

having eliminated z^2 by means of the equation of the surface. The substitution, for the surfaces which have no centre, gives

$$\frac{dy^2}{dx^2}-\left(\frac{y}{x}-\frac{\mathrm{A}''}{\mathrm{A}'}\cdot\frac{x}{y}+\frac{\mathrm{C}''^2(\mathrm{A}''-\mathrm{A}')}{\mathrm{A}'^2\mathrm{A}''xy}\right)\cdot\frac{dy}{dx}-\frac{\mathrm{A}''}{\mathrm{A}'}=0$$

These two equations are of the same form, and may therefore be integrated by the same methods. Let them be represented by the general form

$$\frac{dy^2}{dx^2}-\left(\frac{y}{x}-e\,\frac{x}{y}+\frac{f}{xy}\right)\frac{dy}{dx}-e=0 \quad (\mathrm{A})$$

In which $e=\frac{\mathrm{A}-\mathrm{A}''}{\mathrm{A}-\mathrm{A}'}$ and $f=\frac{\mathrm{A}(\mathrm{A}''-\mathrm{A}')\mathrm{F}}{\mathrm{A}'\mathrm{A}''(\mathrm{A}-\mathrm{A}')}$ for surfaces having a centre, and $e=\frac{\mathrm{A}''}{\mathrm{A}'}$ and $f=\frac{\mathrm{C}''^2(\mathrm{A}''-\mathrm{A}')}{\mathrm{A}'^2\mathrm{A}''}$ for surfaces having no centre.

This might be integrated by resolving it into its factors, but more elegantly by ascending to a differential equation of an higher order. For this purpose let this equation be differentiated. The result is

$$\frac{d^2y}{dx^2}\left(2yx.\frac{dy}{dx}+ex^2-y^2-f\right)+\left(\frac{dy^2}{dx^2}+e\right)\left(x\frac{dy}{dx}-y\right)=0$$

By this and the first equation the constants e and f being eliminated, we find

$$xy\,d^2y+dy\,(xdy-ydx)=0$$

Dividing this by x^2 it becomes

$$\frac{y}{x}d^2y+dy.\,\frac{xdy-ydx}{x^2}=0$$

Let $z=\frac{y}{x}\;\because\;dz=\frac{xdy-ydx}{x^2}$. By this substitution the equation becomes

$$zd^2y+dydz=0$$

$$\because\;\frac{d^2y}{dy}+\frac{dz}{z}=0$$

This being integrated gives

$$zdy=\text{M}\,dx$$

M being an arbitrary constant. Substituting for z its value, the equation becomes

$$ydy=\text{M}xdx.$$

The integral of this is evidently

$$y=\text{M}x^2+\text{N}$$

N being a second arbitrary constant. This is the equation of an ellipse or hyperbola according as $M < 0$ or > 0. It is concentric with the ellipsoid and its axes coincide with those of x and y. Let its axes be a' and b'. The equation expressed in these terms is

$$a'^2y^2 + b'^2x^2 = a'^2b'^2$$

To determine the axes let this equation be differentiated

$$\frac{dy}{dx} = -\frac{b'^2x}{a'^2y}$$

Eliminating y and $\frac{dy}{dx}$ by these equations and (A) the result will be

$$ea'^2 - b'^2 = f$$

Subject to this condition the axes may be of any magnitude. Since b' and a' are the coordinates of the vertices of the angles of the rectangle circumscribed round the axes of each of the ellipses or hyperbolas, which are the projections of the lines of curvature upon the plane xy, it follows that the locus of these points is a curve represented by this equation, b and a' being coordinates measured upon the axes of x and y. Hence it appears that the two equations

$$\begin{aligned} a'^2y^2 + b'^2x^2 &= a'^2b'^2 \\ ea'^2 - b'^2 &= f \end{aligned}$$

determine all the lines of curvature. The first determines the projections of the points of any one line of curvature corresponding to any system of values of a'^2 and b'^2 and the second determines the various systems of values of these last quantities.

If $A' = A''$, $e = 1$ and $f = 0$ and the equation (A) becomes

$$\frac{dy^2}{dx^2}-\left(\frac{y}{x}-\frac{x}{y}\right)\frac{dy}{dx}-1=0$$

The integration is in this case more simply effected by resolving the equation into its factors, which are manifestly

$$\frac{dy}{dx}-\frac{y}{x}=0$$
$$\frac{dy}{dx}+\frac{x}{y}=0$$

These equations being integrated give

$$y=ax$$
$$y^2+x^2=r^2$$

a and r^2 being arbitrary constants. The first, for each value of a, is the equation of a right line through the origin. The second, for each value of r^2, is the equation of a circle of which the origin is the centre. As these constants are not restricted by any condition, it follows that all right lines whatever in the plane xy passing through the origin, and all circles described upon the same plane, with the origin as the centre, are projections of lines of curvature, provided that there are points of the surface corresponding to the points on the plane xy, through which these right lines and circles pass. The condition $A'=A''$ renders the surface, one of revolution round the axis of z. If therefore it be an hyberboloid or paraboloid all circles and right lines whatever thus drawn are actual projections of lines of curvature, since these surfaces have unlimited extent. But if the surface be an ellipsoid, the greatest circle which can be a projection of a line of curvature is the section of the surface by the plane xy itself, which is therefore a line of curvature. In this case, therefore, the real projections of the lines of curvature are the radii of this circle and all circles concentrical with it and included within it.

If the equation of the surfaces which have a centre be expressed in terms of their semiaxes it will be

$$a^2b^2z^2 + a^2c^2y^2 + b^2c^2x^2 = a^2b^2c^2$$

If the surface be an ellipsoid a^2, b^2, c^2, will be all positive.

If it be a double-surfaced hyperboloid a^2 and b^2 are negative, and c^2 positive.

If it be a single-surfaced hyperboloid a^2 and b^2 are positive, and c^2 negative.

The equation of the surfaces which have no centre may be expressed

$$y^2 + mx^2 = pz$$

where m expresses the ratio of the squares of the axes of all elliptic or hyperbolic sections perpendicular to the axis, and p the parameter of the parabolic section of the surface by the plane yz.

Since therefore for the surfaces having a centre

$$\text{A} = a^2b^2 \quad \text{A}' = a^2c^2 \quad \text{A}'' = b^2c^2 \quad \text{F} = -a^2b^2c^2$$

it follows that

$$e = \frac{\text{A}-\text{A}''}{\text{A}-\text{A}'} = \frac{b^2\,(a^2-c^2)}{a^2\,(b^2-c^2)}$$

$$J = \frac{\text{A}(\text{A}''-\text{A}')\,\text{F}}{\text{A}'\text{A}''(\text{A}-\text{A}')} = b^2.\frac{a^2-b^2}{b^2-c^2}$$

For the surfaces which have no centre, $A=0$, $A'=1$, $A''=m$, $C''=-\frac{p}{2}$

$$e=\frac{A''}{A'}=m$$

$$f=\frac{C''^2(A''-A')}{A'^2A''}=\frac{p^2}{4}\cdot\frac{m-1}{m}$$

we shall consider $2a$ the greatest axis and $2c$ the least, in surfaces which have a centre. Also m will be considered greater than unity and p positive. These suppositions, while they simplify the investigation, do not impair its generality.

PROP.

To determine the lines of Curvature of the Ellipsoid.

The equation of the ellipsoid is

$$a^2b^2z^2+a^2c^2y^2+b^2c^2x^2=a^2b^2c^2$$

The quantities e and f are therefore

$$e=\frac{b^2(a^2-c^2)}{a^2(b^2-c^2)}$$

$$f=\frac{b^2(a^2-b^2)}{b^2-c^2}.$$

Since $a>b$ and $b>c$ the quantities e and f are essentially positive.

To determine the lines of curvature it will be necessary to investigate the loci of the equations

$$a'^2y^2 + b'^2x^2 = a'^2b'^2$$
$$ea'^2 - b'^2 = f$$

There are four cases to be considered, according to the different signs the arbitrary constants a'^2 and b'^2 may assume. There are however two of these cases in which one of the loci is impossible. If $a^2 < 0$ and $b'^2 > 0$, the second equation is impossible, for it would equate the sum of two quantities essentially negative with a quantity essentially positive. As therefore the equation of condition between the constants a'^2 and b'^2 cannot under these circumstances be fulfilled, there can be no corresponding lines of curvature. If a'^2 and b'^2 be both negative the first equation is impossible, and therefore there is no corresponding lines of curvature. The only cases therefore which remain to be investigated are where a'^2 and b'^2 are both positive, or b'^2 alone negative.

If a'^2 and b'^2 be both positive the first equation represents an ellipse, and the second an hyperbola. If $b'^2 < 0$ the first represents an hyperbola, and the second an ellipse. The semiaxes of the ellipse and hyperbola represented by the second equation are

$$\mathrm{A}' = a. \frac{\sqrt{a^2 - b^2}}{\sqrt{a^2 - c^2}}$$

$$\mathrm{B}' = b. \frac{\sqrt{a^2 - b^2}}{\sqrt{b^2 - c^2}}$$

With these common axes let an ellipse and hyperbola be described upon the plane xy. With the coordinates of any point of the ellipse as semiaxes, let an hyperbola be described upon the same plane, having its axes coincident with those of the surface, and with the coordinates of any point in the hyperbola as axes, let an ellipse be described in the same manner.. The ellipse and hyperbola thus described will be the projections of lines of curvature of the given ellipsoid, and in the same manner the projections any number of such lines of curvature may be found. The ellipse and hyperbola with the common axes $2A'$ and $2B'$ may be denominated the directrices of the lines of curvature.

The greatest axis of the ellipsoid being $2a$, and the least $2c$, it follows that $a^2-b^2 < a^2-c^2$; and therefore $A' < a$. Hence the common vertices of the directrices fall within the ellipsoid.

Let $ABA'B'$ (Fig 1) be the section of the ellipsoid by the plane of xy, and $CA=a$, $CB=b$. And upon CA assume $ca = A'$ and upon CB produced, $cb = B'$. Let the ellipse $aba'b'$ be described and the hyperbola, Dad, $D'a'd'$, with the same axes. Any point p being assumed in the ellipse $aba'b'$ and its coordinates pm, pm' being drawn, let an hyperbola be described with the lines cm as semitransverse axis and cm' as semiconjugate axis. The branches of this hyperbola will be the projections of two lines of curvature Let tangents be drawn through the vertices A,A' of the elliptic section, and any point P on the hyperbolic directrix, between these tangents Tt and $T't'$ be assumed. The coordinates PM, PM' of this point being drawn, let an ellipse be described with CM and CM' as axes. This ellipse will be the projection of another of the lines of curvature upon the plane xy.

In like manner other points being assumed upon the ellipse $aba'b'$ or hyperbola taT, any number of lines of curvature may be determined. As the point assumed upon the ellipse approaches the vertex b, the coordinate cm diminishes and cm' increases, therefore the transverse axis of the hyperbola continually diminishes, and the conjugate axis continually approaches to equality with bb'. When the vertex b itself is the assumed point, the transverse axis vanishes, and the conjugate axis becomes equal to *bb'*, with which both branches of the hyperbola coincide. Hence the axis *bb'* is the projection of one of the lines of curvature, which must therefore be the section of the surface by the plane *yz*.

As the assumed point approaches the vertex *a*, *cm* increases and *cm'* diminishes, therefore the transverse axis of the corresponding hyperbola continually increases, and the conjugate axis diminishes. When the assumed point is the vertex *a* itself, the conjugate axis vanishes, the transverse axis is *aa'*, and the branches of the hyperbola closing become coincident with the right lines *a*A and *a'*A'.

As the point P assumed upon the hyperbolic directrix approaches the vertex *a*, the conjugate axis of the corresponding ellipse diminishes as well as the transverse axis. When P coincides with the vertex *a*, the conjugate axis of the ellipse vanishes, and the transverse axis becomes equal to *aa'* with which the ellipse itself coincides. This axis of the surface being therefore the projection of one of the lines of curvature upon the plane *xy*, this line must be the section of the surface by the plane *xz*.

As the point P approaches the tangent T*t*, the axes of the corresponding ellipse both increase. When P coincides with the tangent at T the axes become equal to those of the ellipse ABA'B', with which the ellipse then coincides. This ellipse therefore, the section of the surface by the plane *xy*, is one of the lines of curvature.

When the point P passes the tangent, the corresponding ellipse includes the given surface within it, and therefore cannot be the projection of any line in the surface.

It is plain that the transverse axes of all the hyperbolas which are projections of lines of curvature are less than *aa'*, the common axis of the directrices; and all the transverse axes of the ellipses which are projections of lines of curvature, are greater than the same line *aa′*. Hence it appears that the concavities of all these curves must be presented towards the points *a*, *a′*. The *major limit* of the conjugate axes of the hyperbolas is *cb*, and that of the similar axes of the ellipses is CB. The points *a*, *a′* are the projections of four points of the surface towards which the concavities of all the lines of curvature are turned. These points are upon the section of the surface by the plane *xz*. This section being itself a line of curvature all other lines of curvature which meet it intersect it at right angles. The projections of those which intersect it between the vertices of the axis 2*c* and the points whose projections are *a*, *a′*, are hyperbolas. The projections of those which intersect it beyond these points are ellipses. Thus the lines of curvature perpendicular to the section by the plane *xz* change their species as they pass these points, presenting their concavities on both sides towards these points. These are called *umbilical points* of the surface.

The umbilical points of any surface may be determined by considering that they are those points at which the two lines of curvature interchange their species, and at which they therefore coincide. To find these points it will be only necessary to investigate under what circumstances the roots of the equation which determines the value of $\frac{dy}{dx}$ for the lines of curvature, are equal. In the present case this equation is

$$\frac{dy^2}{dx^2}-\left\{\frac{y}{x}-e\frac{x}{y}+\frac{f}{xy}\right\}\cdot\frac{dy}{dx}-e=0$$

The condition of the equality of the roots is

$$\left(\frac{y}{x}-e\frac{x}{y}+\frac{f}{xy}\right)^2+4e=0 \text{ or } (y^2-ex^2+f)^2+4ex^2y^2=0$$

Since e is a quantity essentially positive, this condition can only be fulfilled by

$$xy = 0$$
$$y^2 - ex^2 + f = 0$$

If $y = 0$ and therefore $x = \sqrt{\frac{f}{e}}$ these are the coordinates of the umbilical points already determined. If $x = 0 \therefore y = \sqrt{-f}$. Since f is positive, this value of y is imaginary. Hence there are but the four umbilical points already determined in the ellipsoid.

The delineation of the projection of the lines of curvature of the ellipsoid upon the plane of the greatest and mean axis is represented in Fig. 2.

To determine the projections of the lines of curvature upon the plane xz, let y be eliminated by the equations

$$a^2b^2z_2 + a^2c^2y^2 + b^2c^2x^2 = a^2b^2c^2$$
$$a'^2y^2 + b'^2x^2 = a'^2b'^2$$

This elimination being effected, and b'^2 eliminated by the equation

$$ea'^2 - b'^2 = f,$$

The result is

$$c^2(a^2 - a'^2)fx^2 + a^2b^2a'^2z^2 = ea^2c^2a'^2(a^2 - a'^2)$$

The process of elimination will be facilitated in this case by considering that

$$b^2 + f = a^2e.$$

The equation thus obtained is that of the projection of the lines of curvature upon the plane xz. The quantity a'^2 is variable, and for each value of it there is a distinct line of curvature. Since it has been already proved that a'^2 can never exceed a^2, it follows that $a^2 — a'^2 > 0$, and therefore that all these projections are ellipses. Let $a''c''$ be the semiaxes of any of them corresponding to any proposed value of a'^2, ∴

$$a''^2 = \frac{ea^2a'^2}{f}$$

$$c''^2 = \frac{ec^2(a^2 — a'^2)}{b^2}$$

Eliminating by these equations the variable quantity a'^2, the result is

$$f c^2a''^2 + a^2b^2c''^2 = ea^4c^2$$

$$\text{or} \quad c^2(a^2—b^2)a''^2 + a^2 (b^2—c^2)\, c''^2 = a^2c^2(a^2—c^2)$$

Considering $a''c''$ as variable coordinates this is the equation of an ellipse, since $a^2 — b^2 > 0$ and $b^2 — c^2 > 0$. This is the elliptic directrix for determining the projections of the lines of curvature upon the plane of the greatest and least axes of the ellipsoid.

This equation might have been derived at once from the equation (page 86) for the projections of the lines of curvature upon the plane xy by changing b into c and *v. v.* The result here however is somewhat different from the former. In the present case the lines of curvature are all projected into ellipses, and there is only an elliptic directrix to determine them. The projections of the lines of curvature are therefore all determined by the equations

$$a''^2z^2 + c''^2x^2 = a''^2c''^2$$

$$c^2(a^2—b^2)\, a''^2 + a^2\, (b^2—c^2)\, c''^2 = a^2c^2\, (a^2—c^2).$$

Let A'' and C'' be the semiaxes of the elliptic directrix

$$A'' = a\frac{\sqrt{a^2-c^2}}{\sqrt{a^2-b^2}}$$

$$C'' = c\frac{\sqrt{a^2-c^2}}{\sqrt{b^2-c^2}}$$

Since a^2-c^2 is greater than either a^2-b^2 or b^2-c^2 it follows that A'' and C'' are greater than a and c respectively. On the axes of x and z assume portions equal to A'' and C'', and with these as axes let an ellipse be described. This ellipse will determine the several ellipses which are the projections of the lines of curvature in the same manner as the hyperbolic directrix determined the projections upon the plane of xy. The limits of these ellipses are those which ultimately coincide with the axes of the elliptic directrix. As c'' diminishes a'' approaches to equality with A'', and the corresponding ellipse continually becomes more eccentric until it flattens into a right line when c'' vanishes. Similar observations apply to the other axis when a'' vanishes. It is obvious that all the ellipses which are projections of the lines of curvature are included within the elliptic directrix.

The elliptical section by the plane xz is itself one of the ellipses given by the equation; for if $a'' = a$ $c'' = c$. Those ellipses in which $c'' < c$ divide the elliptic section into zones in the direction of the axis $2a$, and those in which $c'' > c$ divide into zones in the direction of the axis $2c$. These are therefore the projections of the lines of curvature perpendicular to the former.

If the inscribed lozenge be formed in the elliptic directrix by connecting the extremities of its axes, the four sides of this will be tangents to all the ellipses which are projections of the lines of curvature. For let a'' be eliminated by the equations

$$a''^2z^2 + c''^2x^2 = a''^2c''^2$$
$$fc^2a''^2 + a^2b^2c''^2 = ea^4c^2$$

The result is

$$a^2b^2c''^4 + (fc^2x^2 - a^2b^2z^2 - ea^4c^2)\,c''^2 + ea^4c^2z^2 = 0$$

considering c'' variable, this equation represents all the ellipses which are projections of lines of curvature. To determine the equation of the lines which enclose the space occupied by these ellipses let this equation be differentiated, considering c'' as variable, the result of which is

$$2a^2b^2c''^2 + fc^2x^2 - a^2b^2z^2 - ea^4c^2 = 0$$

Eliminating c'' by this and the former,

$$4ea^6b^2c^2z^2 - (fc^2x^2 - a^2b^2z^2 - ea^4c^2)^2 = 0$$

This can obviously be resolved into the factors

$$2\sqrt{e}\,a^3bcz + fc^2x^2 - a^2b^2z^2 - ea^4c^2 = 0$$
$$2\sqrt{e}\,a^3bcz - fc^2x^2 + a^2b^2z^2 + ea^4c^2 = 0$$

And these again may be resolved into the simple factors

$$\sqrt{f}.\ cx + abz - \sqrt{e}.\ a^2c = 0$$
$$\sqrt{f}.\ cx - abz + \sqrt{e}.\ a^2c = 0$$
$$\sqrt{f}.\ cx + abz + \sqrt{e}.\ a^2c = 0$$
$$-\sqrt{f}.\ cx + abz + \sqrt{e}.\ a^2c = 0$$

These are the equations of the sides of the inscribed lozenge in the directrix.

The section of the surface by xz being itself one of the lines of curvature, this lozenge circumscribes it like the others. The points of contact of the sides of the lozenge with it may be found by eliminating z by their equations. The result is

$$x = \pm \sqrt{\frac{f}{e}}$$

Hence the points of contact are the umbilical points of the ellipsoid.

The delineation of the projections of the lines of curvature of the ellipsoid on the plane of the greatest and least axes, is represented in Fig. 3.

PROP.

To determine the lines of curvature of the double surfaced hyperboloid.

The equation of this curve may be derived from that of the ellipse by changing the signs of a^2 and b^2. It is therefore

$$a^2b^2z^2 - b^2c^2x^2 - a^2c^2y^2 = a^2b^2c^2$$

The quantities in the last proposition expressed by e and f become in this case

$$e = \frac{b^2(a^2 + c^2)}{a^2(b^2 + c^2)}$$

$$f = -\frac{b^2(a^2 - b^2)}{b^2 + c^2}$$

Since $a^2 > b^2$, f is essentially negative. The lines of curvature projected upon the plane xy are represented by the equation

$$a'^2y^2 + b'^2x^2 = a'^2b'^2$$

where a'^2 and b'^2 are connected by the relation

$$ea'^2 - b'^2 = f$$

The semiaxes of the directrices are

$$A' = a\frac{\sqrt{a^2 - b^2}}{\sqrt{a^2 + c^2}}$$

$$B' = b\frac{\sqrt{a^2 - b^2}}{\sqrt{b^2 + c^2}}$$

The transverse axis of the hyperbolic directrix coincides in this case with the axis $2b$. The equation of the elliptic directrix being

$$A'^2y^2 + B'^2x^2 = A'^2B'^2$$

and that of the hyperbolic

$$A'^2y^2 - B'^2x^2 = A'^2B'^2$$

With these common axes let an ellipse and hyperbola be described upon the plane xy. As before with the coordinates of the points of the hyperbola as semiaxes ellipses being described, and with those of the points of the ellipse as semiaxes hyperbolas being described, these ellipses and hyperbolas are the projections of the lines of curvature upon the plane xy.

Let ca, cb, (Fig. 4) be the semiaxes of the directrices, and let any point *p* be assumed upon the ellipse; and with its coordinates *cm* and *cm'* as semiaxes let an hyperbola be described, and a point P being assumed upon the hyperbolic directrix let an ellipse be described with the coordinates *cm* and CM. The ellipse and hyperbola thus described are the projections of lines of curvature, and in the same manner any number of such projections may be drawn.

As the point P recedes from the vertex of the hyperbolic directrix, the ellipses continually increase, and they diminish indefinitely as it approaches that point. When P coincides with *b*, one axis of the ellipse becomes equal to *bb'*, and the other vanishes, and the ellipse coincides with the right line *bb'*. This line being therefore the projection of one of the lines of curvature, the section of the surface by the plane *yz* must be that line.

As the point P passes through *b*, and proceeds in the elliptic arc *bp*, one axis continually diminishes, and the other increases. The projections also change their species and become hyperbolas. The transverse axs of the hyperbola continually diminishes as the point approaches *a*, and vanishes when *p* coincides with *a*. The hyperbola then extends itself into a straight line, and coincides with the axis *aa'*. This being the projection of the section of the surface by the plane *xz*, that line is one of the lines of curvature.

The points *b*, *b'* correspond to four umbilical points upon the hyperboloid. As in the ellipse the lines of curvature change their species in passing through these points, and their concavities are every where turned towards them.

The delineation of the projections of the lines of curvature of the double surfaced hyperboloid upon the plane of its imaginary axes is represented in Fig. 5.

PROP.

To determine the lines of curvature of the single-surfaced Hyperboloid.

The equation of the ellipsoid may be changed into that of this curve by making c^2 negative, by which it becomes

$$a^2b^2z^2 - a^2c^2y^2 - b^2c^2x^2 = -a^2b^2c^2$$

By this condition we have also

$$e = \frac{b^2(a^2+c^2)}{a^2(b^2+c^2)}$$

$$f = \frac{b^2(a^2-b^2)}{b^2+c^2}$$

The equation of the projection of the lines of curvature is the same as in the case of the ellipse, and the quantities a'^2 b'^2 subject to the same condition,

$$a'^2y^2 + b'^2x^2 = a'^2b'^2$$
$$ea'^2 - b'^2 = f.$$

The semiaxes of the directrices are in this case

$$\text{A}' = a.\frac{\sqrt{a^2-b^2}}{\sqrt{a^2+c^2}}$$

$$\text{B}' = b.\frac{\sqrt{a^2-b^2}}{\sqrt{b^2+c^2}}$$

Since $\text{A}' < a$ the vertices of the axis $2\text{A}'$ of the directrices must fall within those of the section of the surface by the plane xy.

Let $ca = A'$, $cb = B'$, (Fig. 6) and let the directrices be described with these lines as axes. Any point p in the elliptic directrix being assumed, and its coordinates drawn, with these, cm and cm', let an hyperbola be described; this will be the projection of one of the lines of curvature. In like manner a point P in the hyperbolic directrix will determine an ellipse, which will also be a projection of a line of curvature. As the point p approaches b the transverse axis of the hyperbola diminishes and vanishes when p coincides with b. At this point the projection of the line of curvature is the axis bb', and therefore the section of the surface by the plane yz is a line of curvature. When p coincides with a, or a' the hyperbola closing its branches, becomes a straight line and coincides with the axis aa'. So that the section of the surface by the plane xz is a line of curvature.

It may be observed generally, that those parts only of the ellipses hyperbolas and right lines thus determined, are actual projections of the lines of curvature, which fall outside the elliptical section of the surface by the plane xy; for the value of z for every point within this elliptical section is impossible. Hence it follows that although at the points a, a' the projections of the lines of curvature change their species, and although all their concavities are turned towards this point, there is no corresponding umbilical point in the surface.

The delineation of the projections of the lines of curvature of the single-surfaced hyperboloid upon the plane of its real axes is represented in Fig. 7.

PROP.

To determine the lines of curvature of the elliptic Paraboloid.

The equation of this surface is

$$y^2+mx^2 = pz$$

in which m is positive and > 1. If $p > 0$, the surface lies on the positive side of the the origin, and if $p < 0$ on the negative side. As this is a mere variety in the position of the surface, and none in its species or properties, it is unnecessary to investigate more than one of the two cases. Let p be therefore assumed positive, and the surface consequently on the positive side of the origin.

The equations which determine the lines of curvature being

$$a'^2y^2 + b'^2x^2 = a'^2b'^2$$
$$ea'^2 - b'^2 = f$$

in which

$$e = m$$
$$f = \frac{p^2}{4} \cdot \frac{m-1}{m}$$

The quantities e and f are essentially positive, and the squares of the semiaxes of the directrices are

$$A'^2 = \frac{p^2}{4} \cdot \frac{m-1}{m^2}$$
$$B'^2 = \frac{p^2}{4} \cdot \frac{m-1}{m}$$

Hence $A'^2 m = B'^2$ and therefore the elliptic directrix is similar to the sections of the surface by the planes perpendicular to the axis, for m is the ratio of the squares of the axes of each of these sections.

Let $ca = A'$, $cb = B'$ (Fig. 1.) and with these semiaxes let an ellipse and hyperbola be described. With the coordinates of each point of the ellipse as semiaxes let hyberbolas be described and with those of each point of the hyperbola let ellipses be described, these curves will be the projections of the lines of curvature as in the case of the ellipsoid. The extremities of the axis aa' of the directrices are the projections of two umbilical points upon the plane xy. These points are therefore in the parabolic section of the surface by the plane xz. Since the values of x for these points are $\pm\frac{p}{2}\sqrt{\frac{m-1}{m}}$ and $y=0$, $\therefore z=\frac{p}{4}\cdot\frac{m-1}{m}$. Hence to find the umbilical points let a section of the paraboloid be made by a plane perpendicular to the axis at the distance $\frac{p}{4}\cdot\frac{m-1}{m}$ from the vertex; this section will be an ellipse, and the umbilical points will be at the extremities of its lesser axis.

The delineation of the projection of the lines of curvature of the elliptic poraboloid upon a plane perpendicular to its axis is similar to that of the ellipse, and is represented in Fig. 2.

Prop.

To determine the lines of Curvature of the Hyperbolic Paraboloid.

The equation of this surface is

$$y^2-mx^2=-pz$$

In this case, as before, m may be considered > 1 and $p > 0$.

$$e=-m$$
$$f=\frac{p^2}{4}\cdot\frac{m+1}{m}$$

The lines of curvature are to be determined by the equations

$$a'^2y^2+b'^2x^2=a'^2b'^2$$
$$ma'^2+b'^2=-\frac{p^2}{4}\cdot\frac{m+1}{m}$$

To fulfil the last equation it is necessary that a'^2 and b'^2 should have different signs. In this case the directrices are therefore two conjugate hyperbolas, the squares of the semiaxes of which are

$$\mathrm{A}'^2 = \frac{p^2}{4} \cdot \frac{m+1}{m^2}$$

$$\mathrm{B}'^2 = \frac{p^2}{4} \cdot \frac{m+1}{m}$$

Since $\mathrm{A}'^2 m = \mathrm{B}'^2$ these hyperbolas are similar to the hyperbolic sections of the paraboloid perpendicular to the axis.

With A' and B' as semiaxes, let conjugate hyperbolas be described. Let $ca = \mathrm{A}'$ (Fig 8.) and $cb = \mathrm{B}'$. The coordinates of any point P in either hyperbola being drawn, let an hyperbola be described with these as semiaxes, the transverse semiaxis being that which coincides with the second axis of the directrix. This will be the projection of one of the lines of curvature. In like manner points assumed in the other hyperbola will determine another system of hyperbolas, which are the projections of the other system of lines of curvature. As the points P approach the vertices a and b, the hyperbolas approach the axes aa' and bb' extending themselves nearer to the axes of x and y, until the actual coincidence of the points P with a and b, when the hyperbolas actually become these lines. Which proves that the two sections of the surface by the planes xz and yz are lines of curvature.

The general delineation of the projection of the lines of curvature of the hyperbolic-paraboloid upon a plane perpendicular to the axis is represented in Fig. 9.

The Section of an Ellipsoid by the Plane of the greatest and mean Axes with the Projections of the Lines of Curvature delineated.

Fig. 2.

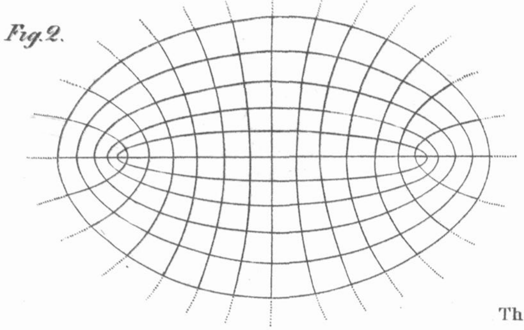
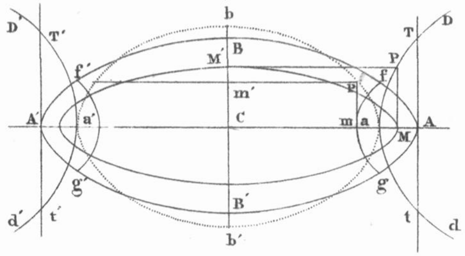
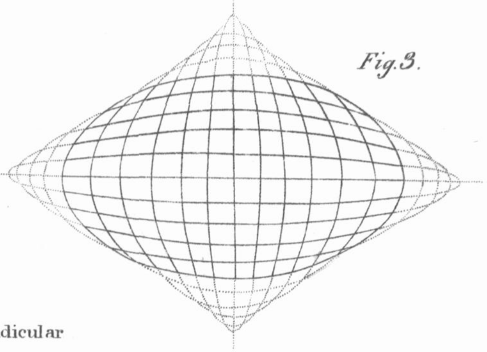


Fig. 1.



The Section of an Ellipsoid by the Plane of the greatest and least Axes with the Projections of the Lines of Curvature delineated.

Fig. 3.



The Section of a double — surfaced Hyperboloid by a Plane perpendicular to its real Axis with the Projections of the Lines of Curvature delineated.

Fig. 4.

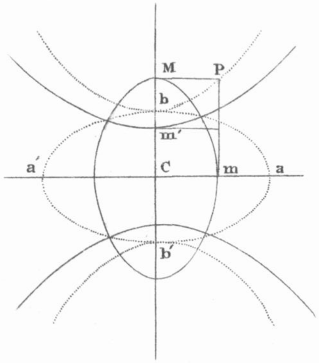


Fig 5.

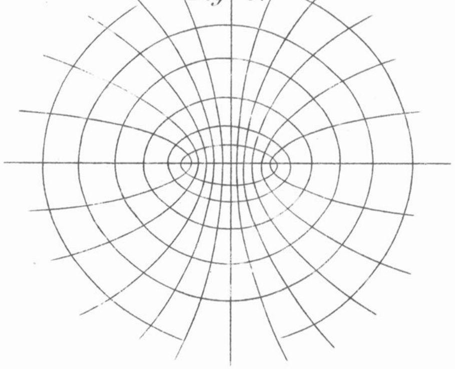
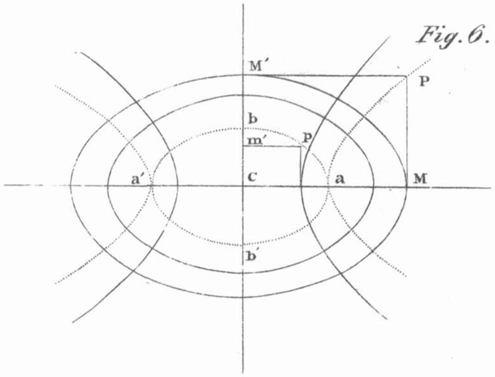


Fig. 6.



The Section of a single – surfaced Hyperboloid by the Plane of its real Axis with the Projections of the Lines of Curvature delineated.

Fig. 7.

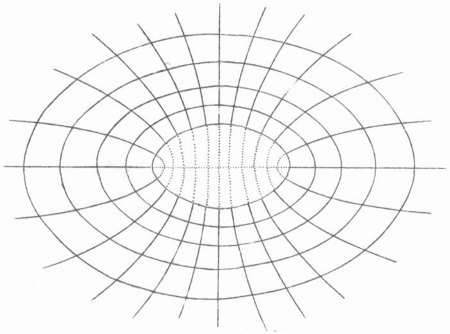
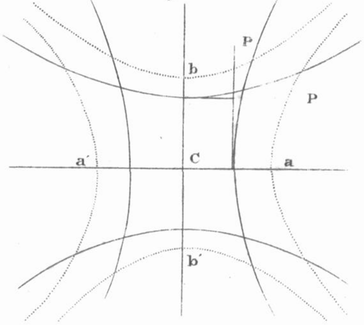


Fig 8.



The Section of an Hyperbolic Paraboloid with the Projections of the Lines of Curvature delineated.

Fig. 9.